# Open problems of the conference Automorphisms of Affine Spaces July 6-10 2009 Radboud University Nijmegen, The Netherlands

(See http://www.math.ru.nl/%7Emaubach/AAS/index.html for more info on this conference).

**Notations:**  $GA_n(\mathbb{C})$  is the set of polynomial automorphisms of  $\mathbb{C}^n$ .

 $\mathrm{Aff}_n(\mathbb{C})$  is the set of affine automorphisms, i.e. compositions of linear maps and translations.

 $\mathrm{TA}_n(\mathbb{C})$  is the subset of tame polynomial automorphisms, i.e. generated by triangular and affine automorphisms.

### Marek Karas

**Problem 1.** Let F, G be Keller maps (i.e. polynomial endomorphisms satisfying det(Jac(F)) = det(Jac(G)) = 1). Suppose

$$F|_{xy=0} = G|_{xy=0}$$

Does this imply F = G?

### Vladimir Bavula

**Problem 2.** Is it true that  $GA_n(k)$  (k a field of char. zero) is generated by tame automorphisms and finitely many one-parameter subgroups of automorphisms?

#### Włodzimierz Danielewski

**Problem 3.** Find an intrinsic algebraic invariant of normal finitely generated k-algebras, that would capture algebraic rigidity of "tubular neighborhood of infinity" and could be thought of as adding a "continuous" dimension to the homotopy type at infinity. This invariant used in a straightforward way, although requiring perhaps complicated calculations, must differentiate all isomorphism classes of the algebras  $k[x, y, z]/(x^nz - y^2 - f(x)y)$ .

**Problem 4.** Classify normal affine equivariant embeddings of connected solvable linear algebraic groups for which all isotropy groups are semisimple. It seems to be easy when maximal tori have dimension one.

# Gene Freudenburg

**Problem 5.** Let  $R = \mathbb{C}[a,b] = \mathbb{C}^{[2]}$ . Let  $D: R[x,y,z] \to R[x,y,z]$  be a locally nilpotent R-derivation which is triangular, i.e.  $D(x) \in R, D(y) \in R[x], D(z) \in R[x,y]$ . Suppose D has a slice. Does it imply that  $\ker(D) = R^{[2]}$ ? Note:  $\ker(D)$  is an  $\mathbb{A}^2$ -fibration over  $\mathbb{A}^2$ .

## Yuriy Bodnarchuk

**Problem 6.** Let k be a field. Is the group  $Aff_n(k)$ , n > 2, a maximal subgroup of  $TA_n(k)$  (tame transformation group)?

## Eric Edo

**Problem 7.** Let  $R := \mathbb{Z}[z]/(z^3)$ ,  $d \in R$ .  $P_1, Q_1 \in R[y]$  such that  $P_1(Q_1(y)) = dy$ . Do there exist  $a, b, c \in R$  and  $P, Q \in R[y]$  such that P(Q(y)) = y and  $P_1(y) = aP(\frac{1}{b}y), Q_2(y) = bQ(\frac{1}{c}y)$ , and a = dc.

# David Wright

**Problem 8.** Let  $G_i$  be the subgroup of  $\operatorname{GA}_n(k)$  that stabilises  $k \oplus kx_1 \oplus \ldots \oplus kx_i$ . Note that  $G_n = \operatorname{Aff}_n(k)$ . If n = 2 then  $G_2 = \operatorname{Aff}_2(k)$ , and  $G_1$  is the set of triangular automorphisms. Now  $\operatorname{GA}_2(k) = G_1 *_{G_1 \cap G_2} G_2$ , the Jung-van der Kulk theorem. Question:  $\operatorname{GA}_n(k) = \langle G_1, \ldots, G_n \rangle$ . And if yes, is

$$TA_n(k) = *G_i$$

the amalgamated product along pairwise intersections?

**Problem 9.** Same question, now for the tame automorphism subgroup: define  $H_i = G_i \cap \text{TA}_n(k)$ . Question:  $\text{TA}_n(k) = \langle H_1, \dots, H_n \rangle$ . And if yes, is

$$TA_n(k) = *H_i$$

the amalgamated product along pairwise intersections?

#### Wenhua Zhao

**Problem 10.** Let  $x = (x_1, x_2, ..., x_n)$  and  $C_n := [0, 1]^{\times n}$ , the *n*-cube in  $\mathbb{R}^n$ . Suppose that  $f(x) \in \mathbb{C}[x]$  and

$$\int_{C_n} f^m(x) \, dx = 0$$

for any  $m \geq 1$ .

Does this imply that f = 0? (Note that, when n = 1, this is true.)

A much weaker version of the problem above is given by the next open problem. But, first let us recall the following notion.

**Definition** Let R be any commutative ring and A a commutative R-algebra. A R-subspace M of A is said to be a Mathieu subspace of A if the following property holds: for any  $a, b \in A$  with  $a^m \in M$  when m >> 0, we have,  $a^m b \in M$  when m >> 0.

Note that, equivalently, one may replace the first "m>>0" in the definition above by " $m\geq 1$ ".

**Problem 11.** Let  $M := \{ f \in \mathbb{C}[x] \mid \int_{C_n} f(x) dx = 0 \}$ . Is M a Mathieu subspace of the polynomial algebra  $\mathbb{C}[x]$ ?

The next open problem asks if Mathieu subspaces are closed under the addition. More precisely,

**Problem 12.** Let  $\mathcal{A}$  be a finitely generated k-algebra, where k is any field. Let  $M_1$  and  $M_2$  be any two Mathieu subspaces of  $\mathcal{A}$ . Is  $M_1 + M_2$  also a Mathieu subspace of  $\mathcal{A}$ ?

For more backgrounds and motivations of the notion of Mathieu subspaces and also the open problems above, see arXiv:0902.0212 [math.CV].

#### Leonid Makar-Limanov

**Problem 13.** Let A,B be commutative rings. How are ML(A), ML(B) and  $ML(A \otimes B)$  related?

**Problem 14.** Let A, B be rings. Suppose that  $A \otimes B = F_n$ , a free associative algebra of rank n. Does it imply that A, B are both free associative algebras too?